FUNDAMENTAL PRINCIPLES OF ALGEBRA

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An eminent mathematician has said recently that of all the high school subjects, algebra has the least and geometry the most educational value. No study in the high school course leaves a more hazy impression on the mind of the average high school student as to its purpose and value than does algebra. The student may put in hours of hard work; he may acquire some skill in performing algebraic operations (to him a highly mechanical accomplishment); he may be able to solve a fairly large number of the problems; he may quote verbatim many definitions, rules and principles; but, when asked what algebra is "all about," what the letters mean, and whether or not there is any "point" or advantage to his accomplishments, the pupil is "at sea." In talking with students, I find that the work done by them, in many cases, is quite purposeless and meaningless. To many the algebra work is done from day to day because it is a task assigned, a sort of daily grind that they must go through, using as their guide-posts the type examples worked out in the algebra texts, or explained by the teacher in the assignment of the lesson.

For some weeks, at the beginning of the present school year, I made it a daily practice to copy on a sheet of paper erroneous statements made in the written work of students in my algebra class. These students had had one year of algebra in the high school and were reviewing in preparation for more advanced work. Samples of the type errors that I noted are:

(1) $x^2 \div x = 1^2$ (3) $(a^2)^3 = a^5$

(2) $\dfrac{a^3 + b^3}{a + b} = a^2 + b^2$ (4) $\dfrac{\cancel{a^2}}{\cancel{a^2} + ab} = ab$

We were reviewing the laws of exponents. In a test I gave the following question: Write illustrations of three laws of exponents, using numbers of arithmetic. You may be interested in some of the responses: (1) $6^2 + 6^3 = 6^5$; (2) $9^4 \div 3^2 = 3^2$; (3) $(4^2)^3 = 4^5$. In speaking of such blunders, one should not omit some which, I am sure, are familiar to all of you:

(1) $\dfrac{\cancel{x} + 1}{\cancel{x}} = 1$ (2) $\dfrac{1}{a} + \dfrac{5}{a} = \dfrac{6}{2a}$ (3) $\sqrt{a^2 + b^2} = a + b$

Pupils will say $\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b}$, when they would never think of saying $\frac{1}{3} + \frac{1}{4} = \frac{1}{7}$. Moreover, they will say $(a + b)^2 = a^2 + b^2$, when they would not say

$$(3 + 4)^2 = 9 + 16, \text{ or } 25.$$

Now my feeling is that the types of errors which I have noted are not peculiarly characteristic of algebra, as taught in any one school. I dare say that many of you are experiencing, or have experienced, similar difficulties. There is something wrong. Where does the trouble lie? Is it to be found in the stupidity of the pupils, or in the difficulty and intangibleness of algebra itself, or does the difficulty come from a lack of emphasis on the outstanding or fundamental notions, which we, as high school teachers, should have in mind, when teaching the subject? I am going to assume that the latter is the principal source of trouble. The question that we have before us then is: What are these fundamental notions that a teacher of high school algebra should have in mind?

Let us imagine that we have here before us a cylindrical vessel the volume of which is to be found. I cut a paper the same size as the bottom and divide it into inch squares. Making due allowance for incomplete squares, I count and find that there are very nearly 22 inch squares in the paper. We shall say that there are exactly 22. If, now, I place in the cylindrical vessel a layer of clay 1 inch thick and just large enough to cover the bottom, I shall have 1 cubic inch of clay standing on each square inch in the bottom. There will be, evidently, 22 cubic inches in the layer altogether. This cylinder is 7 inches high. I pack 7 layers of clay 1 inch thick on top of one another, and the total amount of clay that the cylinder holds is 7 x 22 cubic inches, or 154 cubic inches. If there were, in the base of the cylinder, 17 or 25 or any other number of square inches, and, in the height, 5 or 11 or any other number of linear inches, it is clear that I could calculate the amount of clay the cylinder will hold in the same way, namely, 5×17 cu. in., 11×25 cu. in., and so on. Consequently, we have the following rule for finding the volume of any cylinder: The volume of a cylinder is found by multi-

plying the number of square inches in its base by the number of inches in its height. If the area of the base had been fractional, say 27.1 square inches, each of the layers of clay would have contained 27.1 cubic inches, and, if the height had been 10.4 inches, I could have packed into the cylindrical vessel 10 slabs of clay 1 inch thick and another thinner layer .4 of an inch thick. Then, extending the definition of multiplication so as to include fractional, as well as integral numbers, I could multiply as before, and the rule for finding the volume of the cylinder still holds.

But, suppose it should be necessary to write this rule in a handbook for reference. It would, evidently, take too long and take too much room to write every word as follows: The volume of a cylinder is found by multiplying the number of square inches in the area of its base by the number of inches in its height. There would be no difficulty in knowing the meaning, if one should use abbreviations and write:

$$\text{Vol. of cyl.} = \text{base} \times \text{height.}$$

There are a great many people who constantly need to make use of notes or memoranda of this kind. They are such people as engineers, who need rules for finding the weights that wooden beams and steel bars will support, and shipbuilders and architects, as well as electricians, sailors and military engineers, who use rules for finding the charge of powder needed to blow a breach in a wall of given thickness. Some of the rules they use would take up too much room in their hand-books, even if they should shorten them, as we have the rule for the volume of a cylinder, by using the abbreviation vol. for volume and ht. for height. They find it necessary to use a kind of shorthand for writing their notes. The principle they go by is to use only one letter for a word, such as "height," or a group of words, such as "the volume of a cylinder." An engineer would write the above rule as $V = B \times h$. So far as possible, the letters used are such that they suggest the words for which they stand, V for volume, B for base, and h for height. If it is agreed that two numbers are to be multiplied, when the letters referring to them are written side by side, the rule would be written $v = bh$. That is, the formula $v = bh$ is merely the

shorthand way of writing the rule, the volume of a cylinder is found by multiplying the number of square inches in its base by the number of inches in its height.

The question which I proposed for discussion is: What are the fundamental notions of algebra? So far, it seems that I have been talking about a rule in arithmetic. But in learning this rule in arithmetic, the pupil is beginning the study of algebra. Let us see if I can make clearer what I have in mind. In arithmetic the pupil learns that by the area of a figure is meant the number of square inches, or square units that will cover it, and that the area of a rectangle 7 inches long and 5 inches wide may be found by covering its surface with inch squares, so arranged that they appear in 5 rows each containing 7 inch squares. It is clear to him that the area in question is 5 times 7 square inches. So far, this example is entirely in arithmetic. But let the pupil get away from the particular numbers 5 and 7 and their use in finding the area of the particular rectangle, and try to analyze the process by which he finds the area of any rectangle. The essence of the process is the multiplication of the length of the rectangle by the width. Just at the moment the pupil makes this analysis, he has begun the study of algebra. Even so, the moment a pupil sees that the volume of a cylinder, no matter whether its base is 17, 22, or 25 square inches, or its altitude 5, 7, or 10.4 inches, is equal to its base multiplied by its height, that pupil has crossed the boundary that separates arithmetic from algebra. As stated by Nunn in his *Teaching of Algebra,* "The most fundamental element in algebra is analysis."

Arithmetic and algebra are very closely related. One studies algebra in arithmetic and arithmetic in algebra. It is not an easy matter to define algebra. Neither is it an easy matter to draw a definite line between arithmetic and algebra. In fact, it is impossible to separate two sciences so closely related. For us, the line of demarcation is not an important consideration. The important thing for us, as teachers of algebra, is to realize that the difference between arithmetic and algebra is not so much a difference in the thing studied, as a difference in the way one thinks of the thing studied, whether, as in arithmetic, the manipulation of particular numbers to get a particular

numerical result, or, as in algebra, the process involved in the manipulation to get a definite expression for this process.

There are many types of problems in arithmetic, the analysis of which, yields a rule for solving every problem of the type. In arithmetic, the pupil finds the area of a rectangle, the area of a square and of a circle, as well as the volume of a rectangular solid, of a cube, and of a cylinder. He also solves problems in finding the surface and volume of a prism, and of a pyramid, as well as of a cylinder, a cone and a sphere. In percentage problems the pupil finds the percentage by multiplying the base by the rate per cent; in interest problems, he finds the interest by multiplying the principal by the rate and this product by the time expressed in years. All of these and many more are merely examples of types of problems in arithmetic, which, when analyzed, give rise to definite rules. They are types of problems which may be used to decided advantage in beginning the study of algebra, when it is so necessary that the pupil put definite content into the symbolic expressions that he is learning to manipulate. Let us then set down, as the first fundamental element for the teacher of high school algebra to emphasize, this idea of analysis, the bringing to light of the essential process concealed in the numerical garb of arithmetical problems.

For the study of algebra, as we have seen it thus far, the essential element is the power that every pupil has, at least to some extent, of seeing the process involved in a type of arithmetical problem. A sufficient command of English to state the process, once it has been discovered, is presupposed. But, as we saw in studying the derivation of the formula for the volume of a cylinder, one cannot get on very well in writing the rules that one learns and needs without some kind of abbreviations or symbols for expressing these rules. To write $v = bh$ is a simple to gravity. In fact, without the aid of symbols, many of the matter in comparison with writing the long rule for finding the volume of a cylinder. $d = \frac{1}{2} gt^2$ requires much less time for writing and much less space than the statement: The distance through which a body falls from rest equals ½ the product of the square of the time of falling multiplied by the constant due to gravity. In fact, without the aid of symbols, many of the rules that have been discovered in mathematics could scarcely

be made usable. One writes $(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2.} a^{n-2}b^2 + \cdots$, but when the attempt is made to express in words the rule of which this is the symbolic expression, it is found very lengthy indeed, and almost useless for practical purposes.

Nunn in his book entitled "Exercises in Algebra (including Trigonometry)" gives interesting lists of exercises, for practice in expressing rules and principles in the shorthand of algebra, as well as exercises to give skill in writing formulae for solving certain types of arithmetic problem. With your permission I shall read and discuss a few of these exercises.

(1) Using the shorthand of algebra, write the rule for calculating the cost of a number of things, when you know the price of each. Stated in words the rule is: The cost is obtained by multiplying the price of each thing by the number of things. In the symbols of algebra, $C = np$.

(2) Write the rule, in the shorthand of algebra, for calculating the cost (C) of a certain number of things (N), when you know how much (c) another number of the same kind of things (n) costs. Clearly, that is: $C = N\frac{c}{n}$.

(3) A green-grocer buys a certain number of oranges at a certain price per dozen and sells them at so much each. Write a formula for his profit after selling a certain number. Of course, the profit equals the difference between the selling price and cost of each multiplied by the number of oranges sold. Hence, the formula is: $P = N(s - \frac{c}{12})$.

(4) Using the shorthand of algebra, write the rule for finding the number of spoonfuls of tea required for a certain number of persons according to the receipt, "One spoonful of tea for each person and one for the pot."

(5) The time (t) in hours for cooking a joint of beef of given weight (w) is given by the rule, "Allow a quarter of an hour for every pound and 20 minutes over." Write the rule in algebraic symbols.

(6) Write the formula for finding the weight of a bag containing 37 marbles, given the weight of a bag (b) and the weight of a single marble (m). The formula is: $w = b + 37m$.

(7) Write the formula in exercise 6 for any number of marbles (n). This gives $w = b + nm$.

(8) The length of shelf that would be occupied by 8 books of a certain thickness, followed by 5 books of another thickness is given by a formula. Write the formula. We have $l = 8t_1 + 5t_2$.

(9) Write the formula for the length of shelf that would be occupied by two sets of volumes, given the number and thickness of the volumes in each set. The formula is $l = n_1t_1 + n_2t_2$.

Now, such exercises as these are a revelation to high school pupils, who have never stopped to think what the symbols of algebra really mean. I recommend them for your use in algebra classes.

The history of algebra shows that the growth of a concise symbolism has closely paralleled the development of the science. It may be of help in appreciating the struggle that pupils have in understanding the meaning of the symbols of algebra, to mention the periods through which the world has passed in developing the present method of writing the equation, and to give an illustration, under each period, of the method used.

Three periods are rather clearly marked. The first is the rhetorical period. In the algebra of Mohammed-ben-Musa, written during the rhetorical period, is found the quadratic equation $x^2 + 10x = 39$ written as follows: "One square and ten roots of the same amount to thirty-nine." That is, equations are written entirely in words and no abbreviations or symbols are used. To Mohammed-ben-Musa the equation which I have given meant: What is the square which increased by 10 of its roots makes 39? Incidentally, this may throw some light on the reason for the expression, root of an equation.

The second period in the development of the symbolism of algebra is known as the syncopated period. The words used in writing equations were abbreviated, somewhat as the words were abbreviated in the rule for finding the volume of a cylinder. Diophantus, an Alexandrian, living during this second period, perhaps in the first half of the fourth century, wrote a work in which he represented the unknown quantity in the equation

by the final letter, *s*, of the word used for unknown quantity. He used a letter which looks somewhat like the letter i, for the word equals, it being the final letter of the word.

The third period, which I shall mention, is the period of symbolic algebra. Vieta has the honor of founding symbolic algebra, much the same as we have it today. Before his time (about 1600) unknown quantities were represented by letters, but the powerful influence of using letters for known quantities as well, had its beginning with Vieta. Algebra became a search for operations to be performed, rather than a search for particular values, that is, the idea of function was already beginning to enter the science.

Now, when it took the world until about 1600 to develop the present symbols for writing equations, it is not to be expected that high school pupils will master these symbols, except through well directed effort and hard work. Pupils in the elementary school spend much time during the first two or three years learning to write numbers in figures and in learning to read the numbers so written. It is necessary for them to do this. It is equally necessary for pupils in the high school to have much practice in interpreting the symbols used in algebra, and to extend this practice over a long period.

Perhaps this question arises: Are the x's, a's and b's used in algebra to be thought of as abbreviations for words, or do they stand for numbers? Before answering the question, it is necessary for us to get clearly before us an idea of what algebra, in its broader aspects, means. There is no limit to the number of possible algebras. Wherever a field of inquiry exists, a set of symbols or abbreviations may be devised, and a set of consistent principles may be determined to facilitate the investigation of questions arising in this field. That is, a certain type of algebra may be built up. For example, chemistry has its symbols. It also has its algebra. One writes the identity, $Mg + H_2O = MgO + H_2$, the symbols used being abbreviations for the names of certain chemical elements. This identity means that the "matter" referred to represents itself in two different forms, and such algebraic statements are of distinct aid in solving the problems of chemistry. Another type of algebra that may be of interest is that of George Boole, an English philoso-

pher and mathematician of about 1850. He invented an algebra that has the unique distinction of being the only algebra that has nothing to do with numbers. He denotes space by the symbol i. The letters a, b, c signify definite regions of space. $a + b$ means the portion of space made up of a and b, no portion counted twice. ab means that portion common to a and b. a^1 means space that is not a; thus $a + a^1 = i$. $a + a$, being the region made up of the regions a and a, of course, equals a. That is, $a + a = a$. Moreover, aa, being the region common to a and a, equals a. That is, $aa = a$. No coefficients or exponents are needed in Boole's algebra. Many of you are familiar with Hamilton's quaternions, a type of algebra in which the commutative law of multiplication does not hold. The hope has been expressed that algebras might be devised by means of which theological and political controversies might be settled through calculations with symbols, rather than through denunciations, discussions and debates. Then it would be proper for one politician to say to another, "Let us sit down and calculate." Now, the point which I am trying to make is that the x's, a's, and b's of algebra, considered in this general sense, do not stand for numbers. They stand for words and the words for which they stand may or may not refer to numbers.

In ordinary algebra the words referred to by the symbols are the names of things that have connected with them ideas of number. In this sense, then, the symbols of ordinary algebra refer to numbers; but they refer to numbers only in so far as the words, which they replace, stand for numbers. Much of the difficulty which the beginner has in understanding what algebra is "all about" may be obviated by emphasizing the notion that the symbols of algebra are abbreviations for words. $V = bh$ should mean volume equals base multiplied by height to the beginning pupil. It is hard for a pupil to see how b can stand for a particular number, without standing for this number or that one. It is easy to see that b stands for base, either, 17, 22, 25 or what not.

To sum up our discussion, let us say, that the second fundamental notion for a teacher of high school algebra to emphasize is that the symbols of algebra, highly developed as they are, have

as their purpose the clear, concise, usable expression of results of analysis, the symbols being abbreviations of the words used in expressing these results.

As was seen in the oft cited formula for the volume of a cylinder, the words and phrases were replaced by letters with a gain, both in the conciseness of the statement of the rule and in the clearness of its expression. Then, too, this formula, $V = bh$, may be made to yield other useful formulas, $b = \frac{V}{h}$ and $h = \frac{V}{b}$, if one merely applies to it the principle that, if the product of two factors and one of the factors are known, the second factor may be found by dividing the product by the known factor. Now the aim and end of all algebras, whether the algebra of chemistry, the algebra of political reformers, or the algebra of numbers is the same. It is to correct the weakness of language in expressing clearly and concisely the thoughts and ideas used in various fields of investigation, by expressing these thoughts and ideas in symbolic language. Yes, it is more than this. It is to so develop the symbols used, that they may be manipulated, conveniently, according to firmly established principles, to the end that they may be of aid in the investigation of questions arising in the different fields. It is just here, in my opinion, that the algebra used at present in the discussion of problems arising in educational tests and measurements is very deficient.

The symbols of ordinary algebra have a distinct advantage over the symbols of chemistry and of many other algebras in that they are manipulated according to the same principles as those used in arithmetic. As an example, let us consider the problem of squaring the mixed number, $3\frac{1}{2}$. The arithmetical work of solving this problem is carried out as follows: $3 \times 3 + 3 \times \frac{1}{2} + \frac{1}{2} \times 3 + \frac{1}{2} \times \frac{1}{2}$ and this may be written $3^2 + 2 \times 3 \times \frac{1}{2} + (\frac{1}{2})^2$. Analyzing the process of squaring $3\frac{1}{2}$, one sees that the form of the solution is quite independent of the particular number and fraction chosen for the problem. The multiplication is performed in the same way with any integral number and fraction. Stating the result of the analysis in words one has: The square of a number plus a fraction equals the square

of the number, plus the product of the number and the fraction, plus the product of the fraction and the number, plus the fraction squared. This is the same as: The sum of the square of the number, twice the product of the number and the fraction and the square of the fraction. If the words of the last two statements be abbreviated by the symbols of algebra, we have:

$$(n + f)^2 = nn + nf + nf + ff = n^2 + 2nf + f^2.$$

But now we notice that the work might have been much abbreviated, if we had dealt with the symbols, n and f, just as if they were figures. The symbols are of such a character that they correspond precisely to the figures used in the arithmetical calculation. It is not necessary to go through the process with the figures first, and afterwards to express the rule resulting from the analysis in symbols. The letters may be manipulated, as if they were figures, with just as much certainty of arriving at a true result.

The example given falls under the class of equalities called identities. An identity is essentially a declarative sentence. It is true, no matter what numbers are referred to by the symbols. In high school algebra and in mathematics beyond the high school much more use is made of identical transformations than the time spent in teaching the subject would indicate. It is an open question as to whether identities are not more important, both in high school algebra and in mathematics beyond the high school, than equations.

The principles used in writing identical expressions are simple and few in number: (1) Multiplying and dividing both numerator and denominator of a fraction by a number does not alter the value of the fraction. (2) Adding and subtracting the same number does not change the value of an expression. (3) Multiplying and dividing by the same number does not change the value of an expression. (4) Indicated processes may be performed, or processes to be performed may be indicated in an equivalent way, without changing the value of an expression. In teaching identities every step taken in the manipulation of the symbols should be proved by reference to some one of these principles. Then, perhaps, the error of subtracting the same

number from numerator and denominator of a fraction, and assuming that the value of the fraction remains unchanged, would not be so common among pupils.

I wish to discuss another example, under the general topic of the manipulation of symbols, somewhat different from the one already discussed. Let it be required that one find the divisor in a division problem, knowing that the dividend is 27, the quotient 4, and the remainder 3. The arithmetical reasoning involved is somewhat as follows: Since 4 times the divisor added to 3 gives 27, 4 times the divisor must be 3 less than 27, or 24. Then the divisor is 6. Writing the steps one has:

$$4 \times \text{the divisor} + 3 = 27$$
$$4 \times \text{the divisor} = 27 - 3, \text{ or } 24$$
$$\text{The divisor} = 6$$

That is, an equation has been solved completely.

The analysis of this particular equation indicates that whatever the dividend, quotient and remainder, the reasoning involved in finding the divisor is the same. If the essential features of the analysis be stated in words, and finally, expressed in symbols, one has:

$$qd + r = D$$
$$qd = D - r$$
$$d = \frac{D - r}{q}$$

Thus, in the case of the formula, one notices that the letters are used precisely as figures in arithmetic, according to the same principles, and with equal certainty of a true result. This, no doubt, is the reason for the feeling on the part of so many pupils that the symbols of algebra stand for numbers, rather than for words. Furthermore, it should be said that by manipulating the symbols of a formula according to the laws of arithmetic the mathematician may obtain results which lead to discoveries entirely unknown to the skilled electrician or mechanical engineer.

The equation differs from the identity in that it is, essentially, an interrogative sentence. Its members are equal for only certain values of the letters involved, and these are the numbers asked for by the equation. The process of finding these particular values is called solving the equation and the steps in the

solution are, as in the case of the identity, governed by principles in arithmetic. It is a lamentable fact that high school pupils do not know and understand the principles they use in solving equations, and in making identical transformations. They multiply such an expression as $\frac{x^2}{3} + \frac{2x}{3} - 5$ by 3 in order to simplify it, and assume that the resulting expression is equal to the first. They speak of the value of an equation, and say that multiplying each member of an equation by the same number does not change the value of the equation. Our pupils need to be clearer on the use of the following principles:

(1) If the same number be added to equals, the sums are equal.

(2) If the same number be subtracted from equals, the remainders are equal.

(3) If equals be multiplied by the same number, the products are equal.

(4) If equals be divided by equals, the quotients are equal.

(5) Multiplying or dividing the members of an equation by zero is not permissable.

Huntington in his article on "Fundamental Propositions of Algebra" says: "Until recently, high school algebra has been taught largely as a collection of rules for the manipulation of algebraic symbols. It has not at all been the developed science that elementary geometry has long since become. In fact, if it were not for the study of plane geometry in our high schools, it is doubtful whether or not our pupils would ever get from the study of algebra alone any clear notion as to what is meant by a mathematical demonstration, and yet algebra is better suited than geometry to show what is essentially involved in mathematical reasoning."

Now, having taken as the first fundamental element in algebra, the analysis of type problems in arithmetic, and as the second, a concise, usable symbolism for expressing the essentials of the analysis, we are ready to state a third fundamental notion that the teacher of high school algebra should emphasize. It is this: The symbols of algebra may be "exercised" like the figures in arithmetic and according to the same principles.

There is a fourth fundamental element which should be mentioned. The pupil in arithmetic, sooner or later, reaches the point, where, in his work in division, the dividend is not exactly divisible by the divisor. A new kind of number, called the fraction, is the result. Just so, in algebra, there comes a time when subtraction is no longer possible, and the necessity arises for a new kind of number, the negative number. Again, in arithmetic, numbers are found such that their square roots cannot be obtained, an absurd situation, and hence we have numbers called surds. On the other hand, in algebra, the pupil attempts to find the square root of -4, and comes in contact with another strange number called an imaginary.

Thus we have, as Huntington so well points out, not one science of algebra, but rather a collection of closely related sciences. We have the algebra of positive integers; the algebra of all integers, positive, negative, and zero; the algebra of positive rationals; the algebra of all rationals; the algebra of all real quantities, rational, irrational, positive, negative, and zero; and finally the algebra which includes all of the others, the algebra of complex quantities.

$4 \times 5 = 20$ is called multiplication. Also, $\frac{1}{2}$ of $10 = 5$ is called multiplication, although the pupil insists that, in the latter case, the number is divided. Again, in the problem of finding the compound amount of \$1.00 for 5 years at 4 per cent, one writes $(1.04)^5 \times \$1.00$. But, suppose that the time is $5\frac{1}{2}$ years. The amount at the end of $5\frac{1}{2}$ years cannot be found by $5\frac{1}{2}$ multiplications. It seems reasonable, however, that the process should be indicated by the same symbolism, namely $(1.04)^n \times \$1.00$, whatever this process may be.

The symbolism of algebra, originally invented to express the simple operations in arithmetic, is found so convenient, when numbers beyond positive integers are made necessary, that the definitions of these simple operations are deliberately made to include the less simple operations with the new kinds of numbers. This element of the gradual extention of the number concept and the need for a corresponding change in the definitions of algebraic operations is the fourth fundamental notion which I wish to point out.

I have tried to point out four fundamental elements which a teacher of high school algebra should have in mind:

(1) The ability to see the essential process in an arithmetical problem, that is, analysis.

(2) The clear, concise, usable expression in symbols of the rule resulting from the analysis.

(3) The manipulation of the symbols of ordinary algebra according to the principles of arithmetic.

(4) The extension of the number concept, and the need for a corresponding change in the definitions of fundamental operations.

In closing, I wish to acknowledge my indebtedness, for many of the ideas I have tried to make clear, to T. Percy Nunn, whose book on *The Teaching of Algebra* has been of great help to me.